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Space-Time Microstructure and Singularity-Free Quantum Field Theory†

GERALD ROSEN

Drexel University, Philadelphia, Pennsylvania 19104

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Abstract

It is shown that the dynamical consistency requirements of quantum field theory and the Lorentz-invariant character of particle kinematics and wave equations are compatible with the postulate that physical space is a complex manifold with Euclidean–Gaussian measure in the small. Such a postulate for the microstructure of space introduces a fundamental length $\lambda(\gtrsim 10^{-16}$ cm) and leads to Δ-functions that are analytic on the light-cone for a free field, and hence to self-energies and renormalization constants that are finite for interacting fields.

It has been realized for many years that there is no compelling reason to assume a Minkowskian metric structure for space-time down to arbitrarily small distances (Pauli, 1933; Heisenberg, 1938, 1950; Blokhintsev, 1964, 1965) (less than about 10^{-16} cm), and recent authors (Réidi, 1967; Lundberg & Réidi, 1968; Dardo *et al.*, 1969) have considered the experimental observation of processes at very high energies that might reveal small-scale inhomogeneity and nonisotropy of space-time. Viewed in a basic theoretical way, a departure from Minkowskian metric structure in the small must admit a formalism that satisfies the rather stringent dynamical consistency requirements of quantum field theory and is consonant with the experimentally established Lorentz-invariant character of particle kinematics and wave equations. The purpose of the present note is to describe a simple departure from Minkowskian metric structure that has the latter requisite features. Moreover, the space-time microstructure examined here leads to Δ-functions that are analytic on the light-cone for a free field, and hence to self-energies and renormalization constants that are finite for interacting fields.

The assumption that space-time is Minkowskian down to arbitrarily small distances can be replaced by the following postulate: *In a class of preferred inertial frames of reference, physical space is a* 3-*dimensional complex manifold with the Euclidean-Gaussian measure*

$$dV(\mathbf{z}) \equiv d^3x\,.(\pi^{-3/2}\lambda^{-3}\,e^{-|\mathbf{y}|^2/\lambda^2}\,d^3y) \qquad (1)$$

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about the point with complex cartesian coordinates $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where λ *is a fundamental length constant.* To illustrate that this postulate is compatible with the dynamical consistency requirements of quantum field theory and the Lorentz-invariant character of particle kinematics and wave equations, we consider neutral scalar meson theory. The field $\phi(x) = \phi(x)^\dagger$ is an hermitian operator for real space-time coordinates $x = (x^0, \mathbf{x})$, and thus we have $\phi(z)^\dagger = \phi(z^*)$ for complex spatial coordinates with $z = (x^0, \mathbf{x} + i\mathbf{y})$. In a preference inertial frame the hermitian Lagrangian is thus

$$L \equiv -\tfrac{1}{2} \int (\partial^\mu \phi(z^*)\, \partial_\mu \phi(z) + m^2 \phi(z^*)\, \phi(z))\, dV(\mathbf{z}) \tag{2}$$

where the infinitesimal spatial volume element is prescribed by (1). Expanding the field in a Taylor series in $\mathbf{y}$ and performing the $\mathbf{y}$-integration termwise, we obtain

$$L = -\tfrac{1}{2} \int (\partial^\mu \phi(x)\, e^{-\lambda^2 \nabla^2}\, \partial_\mu \phi(x) + m^2 \phi(x)\, e^{-\lambda^2 \nabla^2} \phi(x))\, d^3 x \tag{3}$$

in which ∇^2 denotes the 3-dimensional Laplacian operator. The relativistic Klein–Gordon equation

$$(\partial^\mu \partial_\mu - m^2)\, \phi(x) = 0 \tag{4}$$

follows from (3), but for the Hamiltonian we have

$$H = \tfrac{1}{2} \int (\pi(x)\, e^{\lambda^2 \nabla^2} \pi(x) + \phi(x)(m^2 - \nabla^2)\, e^{-\lambda^2 \nabla^2} \phi(x))\, d^3 x \tag{5}$$

with the momentum density

$$\pi(x) = e^{-\lambda^2 \nabla^2}\, \partial_0 \phi(x) \tag{6}$$

It is readily verified that the Heisenberg equations of motion for $\partial_0 \phi(x)$ and $\partial_0 \pi(x)$, generated by (5) and the canonical commutation relations

$$\begin{aligned} &[\phi(x), \phi(x')]_{x^0 = x'^0} = 0 = [\pi(x), \pi(x')]_{x^0 = x'^0} \\ &[\phi(x), \pi(x')]_{x^0 = x'^0} = i\delta(\mathbf{x} - \mathbf{x}') \end{aligned} \tag{7}$$

are consistent with (4) and (6). From (6) and (7) we obtain

$$[\phi(x), \partial_0 \phi(x')]_{x^0 = x'^0} = i(8\pi^{3/2} \lambda^3)^{-1} \exp(-|\mathbf{x} - \mathbf{x}'|^2 / 4\lambda^2) \tag{8}$$

and hence the general commutator is $[\phi(x), \phi(x')] = i\Delta(x - x')$ where

$$\Delta(x) = -(2\pi^2)^{-1} \int_0^\infty (\sin[(k^2 + m^2)^{1/2} x^0])(|\mathbf{x}|^{-1} \sin k|\mathbf{x}|)(\exp - \lambda^2 k^2) \cdot \cdot k(k^2 + m^2)^{-1/2}\, dk \tag{9}$$

is an analytic function for all real $\mathbf{x}$, the uniform absolute bound $|\Delta(x)| < (4\pi^2 \lambda^2)^{-1}$ being implied by (9) for all $m^2 \geqslant 0$. Likewise, with the vacuum state defined as usual through $\phi^{(+)}(x)|0\rangle = 0$, the Feynman causal propagator $\Delta_F(x-x') \equiv \langle 0|T(\phi(x)\phi(x'))|0\rangle$ is given by

$$\Delta_F(x) = (2\pi^2)^{-1} \int_0^\infty (\exp[-i(k^2+m^2)^{1/2}|x^0|])(|\mathbf{x}|^{-1} \sin k|\mathbf{x}|)(\exp -\lambda^2 k^2) \cdot \cdot k(k^2+m^2)^{-1/2} dk \tag{10}$$

an analytic uniformly-bounded function for all real $\mathbf{x}$. Since it follows from (4) that the wave function associated with a one-particle state satisfies the Klein–Gordon equation, the particle kinematics is in strict accord with special relativity. However, for interactions involving the field the propagator (10) gives rise to transition rate formulas that deviate from the customary relativistic forms at very high energies ($\gtrsim \lambda^{-1} \gtrsim 100$ BeV) and to self-energies and renormalization constants that are finite.

Similar formulas are obtained with complex spatial coordinates and the Euclidean–Gaussian measure (1) for quantum electrodynamics. In the interaction picture and Lorentz gauge, the vector potential for the electromagnetic field satisfies the commutation relation $[A_\mu(x), A_\nu(x')] = ig_{\mu\nu}D(x-x')$ where $D(x)$ follows from (9) by putting $m = 0$,

$$D(x) = (8\pi^{3/2}\lambda|\mathbf{x}|)^{-1}[(\exp -(x^0+|\mathbf{x}|)^2/4\lambda^2) - (\exp -(x^0-|\mathbf{x}|)^2/4\lambda^2)] \tag{11}$$

while the anticommutator of the electron field is $\{\psi(x), \bar{\psi}(x')\} = -i(\gamma^\mu \partial_\mu - m) \cdot \Delta(x-x')$ with m set equal to the electron mass in (9). The hermitian interaction Lagrangian is prescribed unambiguously as

$$L_{\text{int}} \equiv -\frac{ie}{2} \int \bar{\psi}(z^*)\gamma^\mu \psi(z)(A_\mu(z) + A_\mu(z^*))\, dV(\mathbf{z})$$

$$= -\frac{ie}{2} \int \bar{\psi}(x)\gamma^\mu(\mathrm{e}^{-\lambda^2\nabla^2} A_\mu(x) + A_\mu(x)\,\mathrm{e}^{-\lambda^2\nabla^2})\,\psi(x)\, d^3x \tag{12}$$

With self-energies and renormalization constants finite to all orders of perturbation theory, the observable predictions of the theory do not conflict with existing experiments (Alvensleben *et al.*, 1968) for a value of $\lambda \lesssim 10^{-16}$ cm.

In summary, the postulate that physical space is a complex manifold with Euclidean–Gaussian measure in the small is not at variance with the dynamical consistency requirements of quantum field theory and the Lorentz-invariant character of particle kinematics and wave equations. It is noteworthy that a modified version of the postulate, with the (normally-distributed random-variable) Gaussian weight function for $\mathbf{y}$ in (1) replaced by some other form, would likewise be compatible with the established features of quantum field theory. Although many extensions of quantum field theory are possible if Lorentz invariance is relaxed in the small, the geometrical theory described here is particularly simple and gives rise to a remarkably neat modification of the existing formalism.

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